Available at
www.ComputerScienceWeb.com
POWERED BY SCIENCE @ DIRECT®

Theoretical Computer Science 301 (2003) 79–101

Theoretical Computer Science

www.elsevier.com/locate/tcs

Undecidable properties of monoids with word problem solvable in linear time. Part II— cross sections and homological and homotopical finiteness conditions[☆]

Masashi Katsura[a], Yuji Kobayashi[b], Friedrich Otto[c,*,1]

[a] *Department of Mathematics, Kyoto-Sangyo University, Kyoto 603-8555, Japan*
[b] *Department of Information Science, Toho University, Funabashi 274-8510, Japan*
[c] *Fachbereich Mathematik/Informatik, Universität Kassel, 34109 Kassel, Germany*

Received 16 February 2001; received in revised form 16 June 2002; accepted 28 June 2002
Communicated by Z. Esik

Abstract

Using a particular simulation of single-tape Turing machines by finite string-rewriting systems the first two authors have shown that all linear Markov properties are undecidable for the class of finitely presented monoids with linear-time decidable word problem. Expanding on this construction it is shown here that also many properties that are not known to be linear Markov properties are undecidable for this class of monoids. These properties include the existence of context-free or regular cross-sections, the existence of finite convergent presentations, the property of being automatic, and the homological and homotopical finiteness properties left- and right-FP$_n$ ($n \geqslant 3$), FHT, and FDT.

[☆] The results of this paper have been presented at the 11th Annual International Symposium on Algorithms and Computation (ISAAC 2000) at Taipei, Taiwan, December 2000.

* Corresponding author.

E-mail addresses: katura@ksuvx0.kyoto-su.ac.jp (M. Katsura), kobayasi@is.sci.toho-u.ac.jp (Y. Kobayashi), otto@theory.informatik.uni-kassel.de (F. Otto).

[1] The work presented here was completed while the third author was visiting at the Department of Information Science of Toho University. He gratefully acknowledges the hospitality of the Faculty of Science of Toho University.

PII: S0304-3975(02)00570-4

1. Introduction

A finitely presented monoid M is given through a finite alphabet Σ, that is, a finite set of *generators*, and a finite string-rewriting system S on Σ, that is, a finite set of *defining relations*. Although M is defined through a finite set of data, many algebraic properties of M are undecidable in general. In fact, Markov established a large class of properties of monoids, nowadays known as *Markov properties*, and proved that, if P is such a property, then it is undecidable in general whether a given finitely presented monoid has property P [19]. In his proof Markov uses a finitely presented monoid with an undecidable word problem at a central point. It follows that his undecidability result only applies to classes of monoids containing monoids with undecidable word problem.

Sattler-Klein [28] extended this result by showing that some Markov properties remain undecidable even for the class of finitely presented monoids with word problem decidable in *polynomial time*. Actually, for each recursively enumerable language L, she constructs a family $\{M_w \mid w \in \Gamma^*\}$ of finitely presented monoids satisfying the following properties: each monoid M_w has word problem decidable in polynomial time; M_w is trivial if $w \in L$; on the other hand, M_w is infinite, non-commutative, non-free, etc. if $w \notin L$. Later this construction was extended by showing that also the homotopical finiteness condition FDT [23] and the homological finiteness conditions left- and right-FP_n $(n \geqslant 3)$ and left- and right-FP_∞ are undecidable for this class of monoids [8].

In [13] the first two authors improved upon Sattler-Klein's result. They consider the class $\mathscr{C}_{\text{lin}}$ of all finitely presented monoids with word problem decidable in *linear time*, and they present a construction that is uniform in that it applies to all linear Markov properties. Here a property P of monoids is called a *linear Markov property* if there are two monoids M_1 and M_2 in $\mathscr{C}_{\text{lin}}$ such that M_1 has property P, while M_2 cannot be embedded into any monoid from $\mathscr{C}_{\text{lin}}$ that has property P. It is shown in [13] that all linear Markov properties are undecidable for the class $\mathscr{C}_{\text{lin}}$. This improves upon Sattler-Klein's result in three ways: the class of monoids considered is further restricted by pushing the time bound for the word problem from polynomial time down to linear time, the result is more general in that it covers all linear Markov properties, and the given proof is uniform, while the construction of [28] has to be adjusted to the particular Markov property considered.

Here we derive additional undecidability results for the class $\mathscr{C}_{\text{lin}}$ by building upon the construction of [13]. In fact, we derive two main results. The first one shows that all those properties are undecidable for the class $\mathscr{C}_{\text{lin}}$ that imply the existence of a context-free cross-section for the monoid considered. Examples of such properties are the property of admitting a regular cross-section, the property of having a finite convergent presentation, and the property of being automatic [9,11]. Observe that all these properties are Markov properties, as each of them implies the solvability of the word problem, but it is not known whether any of them is a linear Markov property. This result (Theorem 3.1) is obtained by combining the construction of [13] with a particular finitely presented example monoid that does not admit a context-free cross-section. Actually this example monoid is taken from [22].

The second main result states that all those properties are undecidable for the class $\mathscr{C}_{\text{lin}}$ that imply the homological finiteness condition left-FP_3 (Theorem 4.1). Examples of such properties are the property left-FP_n for all $n \geqslant 3$, the homological finiteness condition FHT, and the homotopical finiteness condition FDT. It is not known whether these properties are Markov properties. This result is obtained by combining the construction of [13] with a finitely presented monoid considered by Lafont and Prouté in [17,18], and which is shown to be neither left- nor right-FP_3 in [16].

The paper is structured as follows. In Section 2 we define the basic notions concerning monoid presentations and string-rewriting systems in order to establish notation. Further, we restate in short the main properties of the construction from [13] that our proofs are based upon. In Section 3 we derive our first main result concerning cross-sections, and in Section 4 we introduce and discuss in short the various homological and homotopical finiteness conditions mentioned above, and we derive our second main result. In the concluding section we show that the construction of [13] can also be used to prove that the property of having a zero element, which is not a Markov property, is undecidable for the class $\mathscr{C}_{\mathrm{lin}}$.

2. Preliminaries

Here we give the basic definitions that we will need throughout the paper in order to establish notation. For background concerning string-rewriting systems and monoid presentations we refer to the monograph [4].

An *alphabet* is a set Σ of symbols or letters. In this paper we will only consider alphabets that are finite. By Σ^* we denote the set of all strings over Σ including the empty string λ. The concatenation of two strings $u, v \in \Sigma^*$ will simply be written as uv. As this operation is associative, Σ^* is a *monoid* under the concatenation of strings with the identity element λ. In fact, Σ^* is the *free monoid* generated by Σ. For a string $w \in \Sigma^*$, $|w|$ denotes the *length* of w, and $|w|_a$ denotes the *a-length* of w, where $a \in \Sigma$, that is, $|w|_a$ is the number of occurrences of the letter a in the string w.

A *string-rewriting system* on Σ is a set S of pairs of strings from Σ^*. We will consider finite as well as infinite string-rewriting systems. An element of S will usually be written as $\ell \to r$, and it is called a *rewrite rule*. By $\mathrm{dom}(S)$ and by $\mathrm{range}(S)$ we denote the sets

$$\{\ell \in \Sigma^* \mid \exists r \in \Sigma^* : (\ell \to r) \in S\} \quad \text{and} \quad \{r \in \Sigma^* \mid \exists \ell \in \Sigma^* : (\ell \to r) \in S\}$$

of all left-hand sides and all right-hand sides of rules of S, respectively. The system S is called *left-regular* if $\mathrm{dom}(S)$ is a regular language.

A string-rewriting system S on Σ induces several binary relations on Σ^*. The most basic one is the *single-step reduction relation*

$$\to_S := \{(x\ell y, xry) \mid x, y \in \Sigma^*, (\ell \to r) \in S\}.$$

Its reflexive transitive closure $\to_S^*$ is the *reduction relation* induced by S. Further, we are interested in the symmetric closure $\leftrightarrow_S$ of $\to_S$ and its reflexive transitive closure $\leftrightarrow_S^*$. The latter is a congruence relation on Σ^*, and it is called the *Thue congruence* generated by S. By $[w]_S$ we denote the congruence class $[w]_S := \{u \in \Sigma^* \mid u \leftrightarrow_S^* w\}$, and by M_S we denote the factor monoid $\Sigma^* / \leftrightarrow_S^*$ of the free monoid Σ^* by the congruence $\leftrightarrow_S^*$. As M_S is uniquely determined by Σ and S, the ordered pair $(\Sigma; S)$ is called a *monoid presentation*. In fact, if M is a monoid that is isomorphic to M_S, then $(\Sigma; S)$ is called a monoid presentation of M. A monoid M is said to be *finitely presented*, if it has a finite monoid presentation.

The *word problem* for a monoid presentation $(\Sigma; S)$ is the following decision problem:

Instance: Two strings $u, v \in \Sigma^*$.

Question: Does $u \leftrightarrow_S^* v$ hold, that is, do u and v represent the same element of the monoid M_S?

It is well known that there exist finite monoid presentations for which the word problem is undecidable (see, e.g., [4]). Actually, the decidability and even the complexity of the word problem are invariants of finitely generated presentations, that is, if $(\Sigma_1; S_1)$ and $(\Sigma_2; S_2)$ are two presentations of the same monoid, where Σ_1 and Σ_2 are finite alphabets, then the word problem for $(\Sigma_1; S_1)$ is decidable if and only if the word problem for $(\Sigma_2; S_2)$ is decidable, and in case of decidability they both have the same degree of complexity [1]. Thus, we can speak of the decidability and even the complexity of the word problem *for a monoid.*

In this paper we will be concerned with those finitely presented monoids for which the word problem is decidable in linear time. By $\mathscr{C}_{\text{lin}}$ we denote this class of monoids. [2] Thus, if $(\Sigma; S)$ is an arbitrarily chosen finite presentation for a monoid from this class, then there exist a constant $c > 0$ and an algorithm (more specifically, a multi-tape Turing machine) that, given two strings u and v from Σ^* as input, will correctly determine in time $c \cdot (|u| + |v|)$ whether or not $u \leftrightarrow_S^* v$ holds.

Throughout this paper we will be dealing with string-rewriting systems that satisfy certain restrictions. A string $u \in \Sigma^*$ is called *reducible* mod S, if $u \rightarrow_S v$ holds for some string $v \in \Sigma^*$; otherwise, u is called *irreducible* mod S. The set of all reducible strings is denoted by $\mathrm{RED}(S)$, and the set of all irreducible strings is denoted by $\mathrm{IRR}(S)$. Obviously, $\mathrm{RED}(S) = \Sigma^* \cdot \mathrm{dom}(S) \cdot \Sigma^*$ and $\mathrm{IRR}(S) = \Sigma^* \smallsetminus \mathrm{RED}(S)$, and so $\mathrm{RED}(S)$ and $\mathrm{IRR}(S)$ are regular languages, if S is left-regular or finite.

The string-rewriting system S on Σ is called:

- *length-reducing* if $|\ell| > |r|$ holds for each rule $(\ell \rightarrow r) \in S$;
- *weight-reducing* if there exists a weight function $\psi : \Sigma \rightarrow \mathbb{N}_+$ such that $\psi(\ell) > \psi(r)$ holds for each rule $(\ell \rightarrow r) \in S$, where we extend ψ to a morphism $\psi : \Sigma^* \rightarrow \mathbb{N}$ by taking $\psi(\lambda) := 0$ and $\psi(wa) := \psi(w) + \psi(a)$ for all $w \in \Sigma^*$ and $a \in \Sigma$;
- *noetherian* if there is no infinite reduction sequence $w_0 \rightarrow_S w_1 \rightarrow_S w_2 \rightarrow_S \ldots$;

[2] To be more specific, we could fix a countably infinite alphabet Σ_∞ and define $\mathscr{C}_{\text{lin}}$ as the set of all finite presentations $(\Sigma; S)$ satisfying the following conditions:
- Σ is a finite subset of Σ_∞,
- S is a finite string-rewriting system on Σ, and
- the word problem for $(\Sigma; S)$ is decidable in linear time.

However, as the complexity of the word problem is an invariant of finite presentations, we prefer to consider $\mathscr{C}_{\text{lin}}$ as a class of monoids.

- *locally confluent* if, for all $u, v, w \in \Sigma^*$, $u \to_S v$ and $u \to_S w$ imply that v and w have a common *descendant* mod S, that is, $v \to_S^* z$ and $w \to_S^* z$ hold for some $z \in \Sigma^*$;
- *confluent* if, for all $u, v, w \in \Sigma^*$, $u \to_S^* v$ and $u \to_S^* w$ imply that v and w have a common descendant mod S; and
- *convergent* (or *complete*) if it is noetherian and confluent.

Finally, two systems S_1 and S_2 that are defined on the same alphabet Σ are called *equivalent* if they generate the same Thue congruence on Σ^*.

Obviously, the length function $|.|$ is a special weight function, and a weight-reducing string-rewriting system is certainly noetherian. For a noetherian system the properties of confluence and local confluence are equivalent. Further, if S is convergent, then each congruence class $[w]_S$ contains a unique irreducible string $\hat{w} \in \mathrm{IRR}(S)$, and hence, in this case the set $\mathrm{IRR}(S)$ is a complete set of unique representatives for the monoid M_S, that is, a *cross-section*. If S is a convergent system such that the relation $\to_S$ is effectively computable, which is certainly the case for a finite system S, then the word problem for the monoid M_S can simply be solved by computing the irreducible descendants $\hat{u}$ and $\hat{v}$ of u and v, respectively, and by comparing them. In fact, the word problem for a finite convergent system that is weight-reducing can be solved in this way in linear time [3].

In general, it is undecidable whether a finite string-rewriting system is noetherian [12]. On the other hand, the system S is noetherian, if it is compatible with an admissible, well-founded partial ordering $\geqslant$ on Σ^*. This means that $\ell > r$ holds for each rule $(\ell \to r)$ of S. Here a partial ordering $\geqslant$ on Σ^* is called *admissible*, if $u \geqslant v$ implies that $xuy \geqslant xvy$ holds for all $x, y \in \Sigma^*$, and it is *well-founded*, if there does not exist any infinite strictly decreasing sequence $w_0 > w_1 > w_2 > \cdots$.

In order to verify that a string-rewriting system S is confluent, the *critical pairs* of S are considered. These are defined as follows. Let $(\ell_1 \to r_1), (\ell_2 \to r_2) \in S$ such that
(i) either $\ell_1 = x\ell_2 y$ for some $x, y \in \Sigma^*$,
(ii) or $x\ell_1 = \ell_2 y$ for some $x, y \in \Sigma^*$, $0 < |x| < |\ell_2|$.
Then the pair (r_1, xr_2y) or (xr_1, r_2y), respectively, is called a *critical pair* of S. By $\mathrm{CP}(S)$ we denote the set of all critical pairs of S. A critical pair $(p, q) \in \mathrm{CP}(S)$ is called *resolvable* if p and q have a common descendant mod S; otherwise, it is called *unresolvable*. A noetherian system S is confluent if and only if all its critical pairs are resolvable.

If S has an unresolvable critical pair (p, q), then we can simply add the rule $(\hat{p} \to \hat{q})$ or the rule $(\hat{q} \to \hat{p})$ to S in order to resolve this critical pair, where $\hat{p}$ and $\hat{q}$ denote irreducible descendants of p and q, respectively. Of course, it must be ensured that the extended system is still noetherian. For example, this is easily achieved, if we have an admissible linear ordering that is well-founded and compatible with S.

Unfortunately, each new rule may lead to new unresolvable critical pairs, and hence, the process above, which is the basic form of the well-known *Knuth–Bendix completion procedure* [14], may not terminate. In fact, given a finite string-rewriting system S and an admissible well-founded linear ordering $\geqslant$ as input, this process terminates if and only if there exists a finite system that is convergent, compatible with $\geqslant$, and equivalent to S. In this case such a system is determined. On the other hand, if the completion procedure does not terminate, then it enumerates an infinite system that is convergent, compatible with $\geqslant$, and equivalent to S. In this case there does not exist a finite system that has all these properties.

Finally, a string-rewriting system S is called *interreduced* if $\ell, r \in \mathrm{IRR}(S \setminus \{\ell \to r\})$ hold for each rule $(\ell \to r) \in S$. For each convergent system S there exists an interreduced convergent system that is equivalent to S and that yields the same normal forms.

In the next sections we will repeatedly make use of the following construction presented by the first two authors in [13].

Let L be a recursively enumerable language on some finite alphabet Θ such that L is non-recursive. From a deterministic single-tape Turing machine accepting L and a string $w \in \Theta^*$, a finite presentation $(\Delta; T_w)$ of a monoid N_w is obtained in two steps. First a finite presentation $(\Delta_1; T_{1,w})$ of a monoid $N_{1,w}$ is constructed, where $\Theta \subseteq \Delta_1$, and $T_{1,w}$ simulates the computation of the Turing machine on the input w. The system $T_{1,w}$ does not contain any *special* rules, that is, $\lambda \notin \mathrm{range}(T_{1,w})$, and there is a distinguished letter $O \in \Delta_1 \setminus \Theta$ that represents a *zero element* of the monoid $N_{1,w}$, that is, $Ox \leftrightarrow^*_{T_{1,w}} O \leftrightarrow^*_{T_{1,w}} xO$ hold for all strings $x \in \Delta_1^*$. In addition, there are two letters H and E in $\Delta_1 \setminus \Theta$ such that $HE \leftrightarrow^*_{T_{1,w}} O$ if $w \in L$, but $HE \not\leftrightarrow^*_{T_{1,w}} O$ if $w \notin L$. In the former case $T_{1,w}$ is equivalent to a finite convergent string-rewriting system $T^{\infty}_{1,w}$, which is obtained by adding finitely many rules of the form $x \to O$ to $T_{1,w}$, while in the latter case $T_{1,w}$ is equivalent to an infinite convergent system $T^{\infty}_{1,w}$, which is obtained by adding an infinite sequence of rules of the form $x \to O$ to $T_{1,w}$. In either case, the normal form of a string $u \in \Delta_1^*$ with respect to $T^{\infty}_{1,w}$ can be computed in linear time ([13, Lemma 4.6]).

In a second step the presentation $(\Delta; T_w)$ of the monoid N_w is obtained from $(\Delta_1; T_{1,w})$ by taking $\Delta := \Delta_1 \cup \{\alpha, \beta, \gamma\}$ and

$$T_w := T_{1,w} \cup \{\alpha HE\beta \to \lambda, \alpha O\beta \to \gamma\} \cup \{x\gamma \to \gamma, \gamma x \to \gamma \mid x \in \Delta\}.$$

Here α, β, and γ are three additional letters. It follows that N_w is the trivial monoid, if $w \in L$, that is, in this situation T_w is equivalent to the finite convergent system $T^{\infty}_w := \{x \to \lambda \mid x \in \Delta\}$. On the other hand, if $w \notin L$, then $N_{1,w}$ is embedded in N_w by the identity mapping on Δ_1^*, T_w is equivalent to the infinite convergent system

$$T^{\infty}_w := T^{\infty}_{1,w} \cup \{\alpha HE\beta \to \lambda, \alpha O\beta \to \gamma\} \cup \{x\gamma \to \gamma, \gamma x \to \gamma \mid x \in \Delta\}$$

and the normal form computation mod T^{∞}_w can be performed in linear time ([13, Lemma 5.1]). Thus, whether $w \in L$ holds or not, the word problem of the finitely presented monoids $N_{1,w}$ and N_w is decidable in linear time, that is, $N_{1,w}$ and N_w belong to the class $\mathscr{C}_{\mathrm{lin}}$.

3. Cross-sections

Let M be a monoid that is given through a presentation of the form $(\Sigma; S)$. A subset $C \subseteq \Sigma^*$ is called a *cross-section* of M, if C contains exactly one element of each congruence class mod S. Here we will establish the following general undecidability result.

Theorem 3.1. *Let P be an invariant property of finitely presented monoids that satisfies the following two conditions*:
(1) *The trivial monoid has property P.*
(2) *Any finitely presented monoid having property P has a context-free cross-section.*
Then it is undecidable in general whether a given finitely presented monoid with linear-time decidable word problem has property P.

Observe that the existence of a context-free cross-section is an invariant property of finitely presented monoids, that is, if $(\Sigma_1; S_1)$ and $(\Sigma_2; S_2)$ are two finite presentations of the same monoid M, then there exists a context-free cross-section $C_1 \subseteq \Sigma_1^*$ if and only if there exists a context-free cross-section $C_2 \subseteq \Sigma_2^*$. This is a direct consequence of the fact that the class of context-free languages is closed under inverse morphisms.

Before providing a proof for the theorem above, we state some applications which can be seen as the main results of this section.

If a monoid M has a finite convergent presentation, or if it has a left-regular convergent presentation, then the set of irreducible strings is a regular cross-section for M. As each regular cross-section is in particular a context-free cross-section, we obtain the following undecidability results from Theorem 3.1.

Corollary 3.2. *For the class of finitely presented monoids with linear-time decidable word problem the following properties are undecidable in general*:
(1) *The existence of a context-free cross-section.*
(2) *The existence of a regular cross-section.*
(3) *The existence of a left-regular convergent presentation.*
(4) *The existence of a finite convergent presentation.*

We want to derive one more consequence of Theorem 3.1. Let $(\Sigma; S)$ be a monoid presentation of a monoid M, and let $C \subseteq \Sigma^*$ be a regular language such that $C \cap [w]_S \neq \emptyset$ for all $w \in \Sigma^*$. The set C is said to be *part of an automatic structure* for the presentation $(\Sigma; S)$, if the following languages $L_=$ and L_a $(a \in \Sigma)$ are regular:

- $L_= := \{ \nu(u, v) \mid u, v \in C,\ u \leftrightarrow_S^* v \}$,
- $L_a := \{ \nu(u, v) \mid u, v \in C,\ ua \leftrightarrow_S^* v \}$.

Here $\nu : \Sigma^* \times \Sigma^* \to \Sigma_\#^*$ is the encoding defined by

$$\nu(a_1 \cdots a_n, b_1 \cdots b_m) := \begin{cases} (a_1, b_1)(a_2, b_2) \cdots (a_n, b_n)(\#, b_{n+1}) \cdots (\#, b_m) & \text{if } n < m, \\ (a_1, b_1)(a_2, b_2) \cdots (a_n, b_n) & \text{if } n = m, \\ (a_1, b_1)(a_2, b_2) \cdots (a_m, b_m)(a_{n+1}, \#) \cdots (a_n, \#) & \text{if } n > m, \end{cases}$$

and $\Sigma_\# := ((\Sigma \cup \{\#\}) \times (\Sigma \cup \{\#\})) \setminus \{(\#, \#)\}$.

The presentation $(\Sigma; S)$ is called *automatic* if there is a regular set $C \subseteq \Sigma^*$ that is part of an automatic structure for $(\Sigma; S)$, and the monoid M is called *automatic* if it has an automatic presentation. Automatic groups have been studied in detail (see [9]), while automatic monoids have only recently attracted attention [6,11,24]. The most basic result on automatic monoids is the fact that their word problems are decidable in quadratic time.

If C is part of an automatic structure for $(\Sigma; S)$, then C contains at least one string from every congruence class mod S. In fact, it can be shown that there exists a regular cross-section $C_1 \subseteq C$ that is also part of an automatic structure for $(\Sigma; S)$ [9]. This in turn implies that each finitely generated presentation of the monoid M that is presented by $(\Sigma; S)$ has a regular cross-section. Thus, we obtain the following undecidability result from Theorem 3.1.

Corollary 3.3. *It is undecidable in general whether a finitely presented monoid with linear-time decidable word problem is automatic.*

It remains to prove Theorem 3.1. For our proof we will use the following particular example monoid that we have considered before in [22].

Let $\Pi := \{a, b, c, \gamma\}$, and let

$$R_K := \{ba \to ab, bc \to aca, cc \to \gamma\} \cup \{x\gamma \to \gamma, \gamma x \to \gamma \mid x \in \Pi\}.$$

By K we denote the monoid presented by $(\Pi; R_K)$. It is shown in [22, Example 6.4], that K has the convergent presentation $(\Pi; R_K^\infty)$, where

$$R_K^\infty := R_K \cup \{a^n c a^n c \to \gamma \mid n \geqslant 1\}$$

and that K does not have a context-free cross-section. Here we will need some additional properties of K that we now derive.

Lemma 3.4. *The word problem for K is decidable in linear time.*

Proof. We will show that the normal form $\hat{w} \in \mathrm{IRR}(R_K^\infty)$ of a string $w \in \Pi^*$ can be computed in linear time. This then implies that the word problem for K is decidable in linear time.

If w contains an occurrence of γ, then $\hat{w} = \gamma$. This can be checked in linear time. So now we can assume that $w \in \{a, b, c\}^*$. If $|w|_c = 0$, then $\hat{w} = a^\alpha b^\beta$, where $\alpha := |w|_a$ and $\beta := |w|_b$. Again $\hat{w}$ can be obtained in linear time.

Finally, assume that $w = w_1 c w_2 c \cdots w_n c w_{n+1}$, where $w_1, \ldots, w_{n+1} \in \{a, b\}^*$. Let $\alpha_i := |w_i|_a$ and $\beta_i := |w_i|_b$, $i = 1, \ldots, n+1$. If $w_i = \lambda$ for some $2 \leqslant i \leqslant n$, then w contains $c w_i c = cc$ as a factor, and hence, $\hat{w} = \gamma$. Otherwise, let

$$\bar{w} := a^{\alpha_1+\beta_1} c a^{\beta_1+\alpha_2+\beta_2} c a^{\beta_2+\alpha_3+\beta_3} c \cdots c a^{\beta_{n-1}+\alpha_n+\beta_n} c a^{\beta_n+\alpha_{n+1}} b^{\beta_{n+1}},$$

which is the result of applying the rules $ba \to ab$ and $bc \to aca$ as long as possible. Of course, the resulting reduction sequence is of quadratic length, but we can determine $\bar{w}$ in linear time from w without actually simulating this reduction sequence. Finally, $\hat{w} = \gamma$ if there exists an index $1 < i \leqslant n$ such that $\beta_{i-2} + \alpha_{i-1} \geqslant \alpha_i + \beta_i$, where we take $\beta_0 := 0$, as in this case the rule $a^{\beta_{i-1}+\alpha_i+\beta_i} c a^{\beta_{i-1}+\alpha_i+\beta_i} c \to \gamma$ is applicable to $\bar{w}$, and otherwise, $\hat{w} = \bar{w}$. Since this condition can be checked in linear time, we see that $\hat{w}$ can be obtained in linear time from w. $\square$

The following key lemma, which implies in particular that the monoid K does not have a context-free cross-section, is proved in [22, Example 6.4], without stating it formally. Therefore, we just give a short outline of its proof.

Lemma 3.5. *Let $C \subseteq \Pi^*$ be a context-free language. If C contains a cross-section for K, then the set $C \cap [\gamma]_{R_K}$ is infinite.*

Proof. Let $C \subseteq \Pi^*$ be a context-free language that contains a cross-section C_1 for the monoid K. We must verify that the intersection $C \cap [\gamma]_{R_K}$ is infinite. To this end we take w_n to denote the string

$$w_n := a^n ca^{n+1} ca^{n+2} ca^{n+3} ca^{n+4} c$$

for each $n \geqslant 1$. As C_1 is a cross-section for K, it contains an element p_n such that $p_n \leftrightarrow^*_{R_K} w_n$. As w_n is irreducible mod R_K^∞, we see that $p_n \rightarrow^*_{R_K^\infty} w_n$, which implies that p_n is of the form

$$p_n = q_1 c q_2 c q_3 c q_4 c q_5 c$$

for some strings $q_1, \ldots, q_5 \in \{a, b\}^+$, where $|q_1| = n$. As C is a context-free language, we can apply Ogden's lemma (see, e.g., [2]) to p_n for sufficiently large n. We mark the first n letters of p_n, that is, we mark the letters of the prefix q_1. Then p_n can be factored as $p_n = uvxyz$ such that:
(1) u, v, and x or x, y, and z each contain at least one marked letter,
(2) vy contains at most n marked letters, and
(3) $uv^m xy^m z \in C$ for all $m \geqslant 0$.
By a detailed case analysis it is shown in [22, Example 6.4] that $uv^m xy^m z \leftrightarrow^*_{R_K} \gamma$ holds for all $m \geqslant 2$. As by (1) $vy \neq \lambda$, we see that $C \cap [\gamma]_{R_K}$ is indeed infinite. □

Let N_w be the finitely presented monoid with presentation $(\Delta; T_w)$ that is constructed in [13] (see the end of Section 2).

We assume that $\Pi \cap \Delta = \{\gamma\}$, and we define a monoid K_w as the 0-direct product of K and N_w, that is, K_w is given through the finite presentation $(\Pi \cup \Delta; R_{K,w})$, where

$$R_{K,w} := R_K \cup T_w \cup \{\sigma\tau \rightarrow \tau\sigma \mid \tau \in \Pi \setminus \{\gamma\}, \sigma \in \Delta \setminus \{\gamma\}\}.$$

As γ serves as a zero for K as well as for N_w, we see that it also is a zero for the monoid K_w.

Lemma 3.6. (a) *If $w \in L$, then K_w is the trivial monoid.*

(b) *If $w \notin L$, then K is embedded into K_w by the identity mapping on Π^*.*

Proof. (a) Let $w \in L$. Then T_w is equivalent to the trivial system $\{x \rightarrow \lambda \mid x \in \Delta\}$. This means that $\gamma \leftrightarrow^*_{T_w} \lambda$, and so $\gamma \leftrightarrow^*_{R_{K,w}} \lambda$. Hence, we obtain $\tau \leftrightarrow^*_{R_{K,w}} \lambda$ for each $\tau \in \Pi \cup \Delta$, and so K_w is the trivial monoid.

(b) Let $w \notin L$. Then T_w is equivalent to the infinite convergent system T_w^∞. Further, R_K is equivalent to the infinite convergent system R_K^∞. Thus, $R_{K,w}$ is equivalent to the infinite system

$$R^\infty_{K,w} := R^\infty_K \cup T^\infty_w \cup \{\sigma\tau \to \tau\sigma \mid \tau \in \Pi \setminus \{\gamma\}, \sigma \in \Delta \setminus \{\gamma\}\}.$$

We claim that this system is convergent, that is, noetherian and confluent.

Claim 1. *The system $R^\infty_{K,w}$ is noetherian.*

Proof. Let $R_\gamma := \{x\gamma \to \gamma, \gamma x \to \gamma \mid x \in \Pi \cup \Delta\}$, and let $\hat{R}^\infty_{K,w} := R^\infty_{K,w} \setminus R_\gamma$. The subsystem R_γ is length-reducing, and therewith noetherian. Hence, an infinite $R^\infty_{K,w}$-reduction sequence contains infinitely many applications of rules from the subsystem $\hat{R}^\infty_{K,w}$. Now observe that, if $w \to_{R_\gamma} w_1 \to_{\hat{R}^\infty_{K,w}} w_2$, then there is some w_1' such that $w \to_{\hat{R}^\infty_{K,w}} w_1' \to_{R_\gamma} w_2$ holds, as no rule of $\hat{R}^\infty_{K,w}$ contains an occurrence of the symbol γ on its left-hand side. Hence, from an infinite $R^\infty_{K,w}$-reduction sequence we obtain an infinite $\hat{R}^\infty_{K,w}$-reduction sequence.

Now assume that the system $\hat{R}^\infty_{K,w}$ is indeed non-noetherian, and that

$$w_0 \to_{\hat{R}^\infty_{K,w}} w_1 \to_{\hat{R}^\infty_{K,w}} \cdots \to_{\hat{R}^\infty_{K,w}} w_i \to_{\hat{R}^\infty_{K,w}} w_{i+1} \to_{\hat{R}^\infty_{K,w}} \cdots$$

is an infinite sequence of reduction steps mod $\hat{R}^\infty_{K,w}$. Let π_1 denote the projection from $(\Pi \cup \Delta)^*$ onto Π^*, and let π_2 denote the projection from $(\Pi \cup \Delta)^*$ onto Δ^*. Further, for each $i \geqslant 0$, let $u_i := \pi_1(w_i)$ and $v_i := \pi_2(w_i)$. Then for each $i \geqslant 0$, one of the following properties is satisfied:
(1) $u_i \to_{\hat{R}^\infty_K} u_{i+1}$ and $v_i = v_{i+1}$ or $v_i = v_i^{(1)}v_i^{(2)}$ and $v_{i+1} = v_i^{(1)}\gamma v_i^{(2)}$, or
(2) $u_i = u_{i+1}$ or $u_i = u_i^{(1)}u_i^{(2)}$ and $u_{i+1} = u_i^{(1)}\gamma u_i^{(2)}$ and $v_i \to_{\hat{T}^\infty_w} v_{i+1}$, or
(3) $u_i = u_{i+1}$ and $v_i = v_{i+1}$,
where $\hat{R}^\infty_K := R^\infty_K \setminus R_\gamma$ and $\hat{T}^\infty_w := T^\infty_w \setminus R_\gamma$.

Case (1) includes the two subcases that a rule of $\hat{R}^\infty_K$ is used that does or does not introduce a new occurrence of the symbol γ. Analogously, case (2) includes the two subcases that a rule of $\hat{T}^\infty_w$ is used that does or does not introduce a new occurrence of the symbol γ. Finally, case (3) corresponds to the situation that $w_i \to_{\hat{R}^\infty_{K,w}} w_{i+1}$ is an application of a commutation rule from the set $\{\sigma\tau \to \tau\sigma \mid \tau \in \Pi \setminus \{\gamma\}, \sigma \in \Delta \setminus \{\gamma\}\}$.

As γ does not occur on the left-hand side of any rule of $\hat{R}^\infty_{K,w}$, we see that, for each step $u_j^{(1)}\gamma u_j^{(2)} \to_{\hat{R}^\infty_K} u_{j+1}^{(1)}\gamma u_{j+1}^{(2)}$, there is a corresponding step $u_j^{(1)}u_j^{(2)} \to_{\hat{R}^\infty_K} u_{j+1}^{(1)}u_{j+1}^{(2)}$, and for each step $v_j^{(1)}\gamma v_j^{(2)} \to_{\hat{T}^\infty_w} v_{j+1}^{(1)}\gamma v_{j+1}^{(2)}$, there is a corresponding step $v_j^{(1)}v_j^{(2)} \to_{\hat{T}^\infty_w} v_{j+1}^{(1)}v_{j+1}^{(2)}$. Hence, if case (1) occurs infinitely many times, then we obtain an infinite $\hat{R}^\infty_K$-reduction sequence starting with u_0, which contradicts the fact that the system R^∞_K is noetherian. Analogously, if case (2) occurs infinitely many times, then we obtain an infinite $\hat{T}^\infty_w$-reduction sequence starting from v_0, which contradicts the fact that the system T^∞_w is noetherian. Thus, from some point on only case (3) occurs, which is impossible as the set of commutation rules is obviously noetherian as well. Hence, we see that the system $\hat{R}^\infty_{K,w}$, and therewith the system $R^\infty_{K,w}$, is noetherian. □

Claim 2. *The system $R^\infty_{K,w}$ is confluent.*

Proof. As $R^{\infty}_{K,w}$ is noetherian, it suffices to verify that all the critical pairs of $R^{\infty}_{K,w}$ resolve. As γ is the only letter that Π and Δ have in common, the critical pairs of $R^{\infty}_{K,w}$ are just the critical pairs of R^{∞}_{K}, the critical pairs of T^{∞}_{w}, and the critical pairs that result from overlapping the rules of R^{∞}_{K} and of T^{∞}_{w} with the commutation rules. As R^{∞}_{K} and T^{∞}_{w} are convergent, the former critical pairs all resolve, and it is easily seen that also the latter pairs resolve. Thus, the system $R^{\infty}_{K,w}$ is confluent. $\square$

By Claims 1 and 2 the system $R^{\infty}_{K,w}$ is convergent. It follows immediately that, for all $u,v\in\Pi^*$, $u\leftrightarrow^*_{R^{\infty}_{K}} v$ if and only if $u\leftrightarrow^*_{R^{\infty}_{K,w}} v$, that is, K is embedded into K_w by the identity mapping on Π^*. $\square$

As above let π_1 and π_2 denote the projections from $(\Pi\cup\Delta)^*$ onto Π^* and Δ^*, respectively. If $w\in L$, then by Lemma 3.6(a) K_w is the trivial monoid, and hence, the word problem for K_w is decidable in linear time, and the set $\{\lambda\}$ is a (context-free) cross-section for K_w. To solve the word problem for K_w efficiently for the case that $w\notin L$, we will make use of the following technical lemma.

Lemma 3.7. *Suppose that $w\notin L$, and let $x\in(\Pi\cup\Delta)^*$. Then the following three statements hold*:
(1) $x\leftrightarrow^*_{R_{K,w}}\pi_1(x)\pi_2(x)$.
(2) $x\leftrightarrow^*_{R_{K,w}}\gamma$ *if and only if* $\pi_1(x)\leftrightarrow^*_{R_K}\gamma$ *or* $\pi_2(x)\leftrightarrow^*_{T_w}\gamma$.
(3) *If* $x\leftrightarrow^*_{R_{K,w}} y$ *for some* $y\in\Pi^*$ *such that* $y\not\leftrightarrow^*_{R_K}\gamma$, *then* $\pi_2(x)\leftrightarrow^*_{T_w}\lambda$.

Proof. (1) As $R_{K,w}$ contains the rules $\{\sigma\tau\to\tau\sigma\mid\tau\in\Pi\smallsetminus\{\gamma\},\sigma\in\Delta\smallsetminus\{\gamma\}\}$, this statement is obvious for all x satisfying $|x|_{\gamma}=0$. On the other hand, if $|x|_{\gamma}>0$, then also $|\pi_1(x)|_{\gamma}>0$ and $|\pi_2(x)|_{\gamma}>0$, and hence, $x\leftrightarrow^*_{R_{K,w}}\gamma\leftrightarrow^*_{R_{K,w}}\pi_1(x)\pi_2(x)$.

(2) If $\pi_1(x)\leftrightarrow^*_{R_K}\gamma$ or $\pi_2(x)\leftrightarrow^*_{T_w}\gamma$, then by (1) it follows that $x\leftrightarrow^*_{R_{K,w}}\gamma$. Conversely, if $x\leftrightarrow^*_{R_{K,w}}\gamma$, then by (1) $\pi_1(x)\pi_2(x)\leftrightarrow^*_{R_{K,w}}\gamma$. Hence, we see that $\pi_1(x)\pi_2(x)\to^*_{R^{\infty}_{K,w}}\gamma$. Thus, either $\pi_1(x)$ or $\pi_2(x)$ contains an occurrence of the symbol γ, or γ is introduced by an application of a rule from R^{∞}_{K} or from T^{∞}_{w}. In either case it follows that $\pi_1(x)\to^*_{R^{\infty}_{K}}\gamma$ or $\pi_2(x)\to^*_{T^{\infty}_{w}}\gamma$, that is, $\pi_1(x)\leftrightarrow^*_{R_K}\gamma$ or $\pi_2(x)\leftrightarrow^*_{T_w}\gamma$ holds.

(3) Assume that $x\leftrightarrow^*_{R_{K,w}} y$ for some $y\in\Pi^*$ satisfying $y\not\leftrightarrow^*_{R_K}\gamma$, that is, we can assume without loss of generality that $y\in\mathrm{IRR}(R^{\infty}_{K})\cap(\Pi\smallsetminus\{\gamma\})^*$. Then y is also irreducible mod $R^{\infty}_{K,w}$, and so by (1) $\pi_1(x)\pi_2(x)\to^*_{R^{\infty}_{K,w}} y$. As $\pi_2(x)\in\Delta^*$ and γ is the only letter that belongs to Π as well as to Δ, it follows that $\pi_2(x)\to^*_{T^{\infty}_{w}}\lambda$, that is, $\pi_2(x)\leftrightarrow^*_{T_w}\lambda$. $\square$

Based on Lemma 3.7 we obtain the following result.

Lemma 3.8. *The word problem for K_w is decidable in linear time.*

Proof. If $w\in L$, then K_w is the trivial monoid. So let us assume that $w\notin L$. By Lemma 3.4 the word problem for K is decidable in linear time, and from [13] we know that the word problem for N_w is decidable in linear time. Thus, by Lemma 3.7(2) it is decidable in linear time whether a string $w\in(\Pi\cup\Delta)^*$ is congruent to γ mod $R_{K,w}$.

Now assume that $x_1, x_2 \in (\Pi \cup \Delta)^*$ are such that neither x_1 nor x_2 is congruent to γ mod $R_{K,w}$. Then neither x_1 nor x_2 nor any of their descendants mod $R^{\infty}_{K,w}$ contain an occurrence of γ. Since $x_1 \leftrightarrow^*_{R_{K,w}} \pi_1(x_1)\pi_2(x_1)$ and $x_2 \leftrightarrow^*_{R_{K,w}} \pi_1(x_2)\pi_2(x_2)$, we see that $x_1 \leftrightarrow^*_{R_{K,w}} x_2$ if and only if $\pi_1(x_1)\pi_2(x_1)$ and $\pi_1(x_2)\pi_2(x_2)$ have a common descendant mod $R^{\infty}_{K,w}$, if and only if $\pi_1(x_1)$ and $\pi_1(x_2)$ have a common descendant mod R^{∞}_K and $\pi_2(x_1)$ and $\pi_2(x_2)$ have a common descendant mod T^{∞}_w, if and only if $\pi_1(x_1) \leftrightarrow^*_{R_K} \pi_1(x_2)$ and $\pi_2(x_1) \leftrightarrow^*_{T_w} \pi_2(x_2)$ both hold. Thus, $x_1 \leftrightarrow^*_{R_{K,w}} x_2$ is decidable in linear time. □

We have already noted that the set $\{\lambda\}$ is a cross-section for the monoid K_w, if $w \in L$. If $w \notin L$, then we have the following contrasting result.

Lemma 3.9. *If $w \notin L$, then K_w does not have a context-free cross-section.*

Proof. Assume that $w \notin L$, but that $C \subseteq (\Pi \cup \Delta)^*$ is a context-free cross-section for K_w. Hence, there exists a unique element $z \in C$ such that $z \leftrightarrow^*_{R_{K,w}} \gamma$. For each string $x \in \Pi^*$ satisfying $x \not\leftrightarrow^*_{R_K} \gamma$, we have $x \not\leftrightarrow^*_{R_{K,w}} \gamma$ by Lemma 3.7(2). Hence, there exists some $y \in C \setminus \{z\}$ such that $x \leftrightarrow^*_{R_{K,w}} y$. From Lemma 3.7(3) it follows that $\pi_2(y) \leftrightarrow^*_{T_w} \lambda$, and further that $\pi_1(y) \leftrightarrow^*_{R_K} x$. Thus, the context-free language

$$C_1 := \pi_1(C \setminus \{z\}) \cup \{\gamma\}$$

contains a cross-section for K, which by Lemma 3.5 implies that $C_1 \cap [\gamma]_{R_K}$ is infinite. Hence, there exists some element $v \in C \setminus \{z\}$ such that $\pi_1(v) \leftrightarrow^*_{R_K} \gamma$, and so

$$v \leftrightarrow^*_{R_{K,w}} \pi_1(v)\pi_2(v) \leftrightarrow^*_{R_{K,w}} \gamma \leftrightarrow^*_{R_{K,w}} z.$$

This, however, contradicts the assumption that C is a cross-section for K_w. □

Combining the technical results above we easily obtain a proof for Theorem 3.1.

Proof of Theorem 3.1. Given a string w, the presentation $(\Pi \cup \Delta; R_{K,w})$ of the monoid K_w can be constructed effectively. By Lemma 3.8 the word problem for K_w is decidable in linear time.

If $w \in L$, then K_w is the trivial monoid (Lemma 3.6(a)), and hence, K_w has property P. If, however, $w \notin L$, then K_w does not have a context-free cross-section (Lemma 3.9), and hence, it does not have property P. Thus, $w \in L$ if and only if K_w has property P. As chosen above the language L is non-recursive. Thus, it is undecidable whether or not K_w has property P. □

4. The property left-FP_3 is undecidable for $\mathscr{C}_{\mathrm{lin}}$

In the second part of this paper we will show that various homological and homotopical finiteness conditions are undecidable for the class $\mathscr{C}_{\mathrm{lin}}$. Actually, these undecidability results are consequences of the following technical result, where left-FP_3 is a particular homological finiteness condition (see below).

Theorem 4.1. *Let P be an invariant property of finitely presented monoids that satisfies the following conditions*:
(1) *Each monoid with a finite convergent presentation has property P.*
(2) *For each finitely presented monoid N, if N has property P, then N is left-FP_3.*
Then it is undecidable in general whether a given finitely presented monoid with linear-time decidable word problem has property P.

Before proving this main result we present the various finiteness conditions and describe in short the relationships between them. For more information and a detailed derivation of this material we refer to the literature (see, e.g., [16,29]).

Let M be a monoid given through a presentation $(\Sigma; S)$. Then by A we denote the *integral monoid ring* $\mathbb{Z}M$ of the monoid M.

An abelian group C is called a *left A-module*, if there exists a *left action* of A on C, it is called a *right A-module*, if there is a *right action* of A on C, and it is an *A-bimodule*, if there are a left action and a right action of A on C such that $(a_1c)a_2 = a_1(ca_2)$ holds for all $a_1, a_2 \in A$ and $c \in C$.

A mapping $\alpha : C_1 \to C_2$, where C_1 and C_2 are A-modules, is an A-module homomorphism if it is compatible with addition and the action(s) of A. A sequence $C_1 \xrightarrow{\alpha} C_2 \xrightarrow{\beta} C_3$ of such homomorphisms is called *exact at C_2* if $\mathrm{im}(\alpha) = \ker(\beta)$ holds. Here $\mathrm{im}(\alpha) := \{c \in C_2 \mid \exists c' \in C_1 : \alpha(c') = c\}$ and $\ker(\beta) := \{c \in C_2 \mid \beta(c) = 0\}$. Finally, a sequence

$$C_1 \xrightarrow{\alpha_1} C_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_n} C_{n+1}$$

is called *exact*, if it is exact at C_i for all $i = 2, \ldots, n$.

The monoid M is said to be *left-FP_k* (*right-FP_k*) for some integer $k \geqslant 1$, if there exist finitely generated free left (right) A-modules C_i and left (right) A-module homomorphisms δ_i such that the sequence

$$C_k \xrightarrow{\delta_k} C_{k-1} \xrightarrow{\delta_{k-1}} \ldots \xrightarrow{\delta_2} C_1 \xrightarrow{\delta_1} A \xrightarrow{\delta_0} \mathbb{Z} \to 0$$

is exact. Here $\mathbb{Z}$ is considered as the trivial left (right) A-module, $A = \mathbb{Z}M$ is also considered as a left (right) A-module, and $\delta_0 : A \to \mathbb{Z}$ is the *augmentation map* that maps an arbitrary element of A of the form $\sum_{j=1}^{n} z_j a_j$ ($z_j \in \mathbb{Z}, a_j \in M$) onto the sum $\sum_{j=1}^{n} z_j$. The monoid M is *left-FP_∞* (*right-FP_∞*) if it is left-FP_k (right-FP_k) for all $k \geqslant 1$. It has been shown by several authors that a finitely presented monoid is left- and right-FP_∞, if it has a finite convergent presentation. See [7] for a survey.

With the monoid presentation $(\Sigma; S)$ an infinite graph $\Gamma(\Sigma; S)$ can be associated that depicts the single-step reduction relation $\to_S$ on Σ^*. The vertices of this graph correspond to the strings from Σ^*, and its edges correspond to applications of rules of S. Accordingly, an edge e leading from $\sigma(e) := u\ell v$ to $\tau(e) := urv$ is denoted by $(u; \ell, r; v)$, where $u, v \in \Sigma^*$, $(\ell \to r) \in S \cup S^{-1}$, and $S^{-1} := \{(r \to \ell) \mid (\ell \to r) \in S\}$, and $e^{-1} := (u; r, \ell; v)$ denotes the inverse edge leading from urv back to $u\ell v$ (see [29] for the details). Observe that the free monoid Σ^* acts from the left and from the right on this graph by the operation of concatenation. By $P(\Gamma(\Sigma; S))$ we denote the set of all paths in this graph, and by $P_+(\Gamma(\Sigma; S))$ we denote the set of all positive paths, where a path p is called *positive* if it only contains edges of the form $(u;\ell,r;v)$ with $(\ell\to r)\in S$. Further, $P^{(2)}(\Gamma(\Sigma;S))$ is the *set of all pairs of parallel paths*, where two paths $p,q\in P(\Gamma(\Sigma;S))$ are called *parallel* if they have the same initial vertex and the same terminal vertex, that is,

$$P^{(2)}(\Gamma(\Sigma;S)) := \{(p,q) \mid p,q \in P(\Gamma(\Sigma;S)), \sigma(p) = \sigma(q), \text{ and } \tau(p) = \tau(q)\}.$$

Of course, the two-sided action of Σ^* on $\Gamma(\Sigma;S)$ carries over to a two-sided action of Σ^* on $P(\Gamma(\Sigma;S))$ and on $P^{(2)}(\Gamma(\Sigma;S))$.

In [29] Squier studied certain subsets of $P^{(2)}(\Gamma(\Sigma;S))$ that he called *homotopy relations*. For each $B\subseteq P^{(2)}(\Gamma(\Sigma;S))$, there is a uniquely determined smallest homotopy relation $\sim_B\ \subseteq P^{(2)}(\Gamma(\Sigma;S))$ that contains B. Now $(\Sigma;S)$ is said to be of *finite derivation type*, FDT for short, if $P^{(2)}(\Gamma(\Sigma;S))$ has a finite *homotopy base*, that is, if there exists a finite set B such that $\sim_B$ is all of $P^{(2)}(\Gamma(\Sigma;S))$. Squier proved that this property is actually an invariant of finitely presented monoids, and that each monoid with a finite convergent presentation has the property FDT. In fact, he proved that the set of critical pairs of a convergent system together with the corresponding resolutions yields a homotopy base for $P^{(2)}(\Gamma(\Sigma;S))$.

Finally, Pride associated with the monoid presentation $(\Sigma;S)$ a certain A-bimodule Π. Actually Π is the first homology group of a 2-complex $\mathscr{D}$ with underlying graph $\Gamma(\Sigma;S)$. Now $(\Sigma;S)$ is said to be of *finite homology type* (FHT), if Π is finitely generated as an A-bimodule [25,30]. Again it turned out that this is an invariant of finitely presented monoids. In fact, Π is embedded in the free A-bimodule $A\cdot S\cdot A$ generated by S [10]. Here a set of formal generators $\{[e]\mid e\in S\}$ is chosen that is in one-to-one correspondence to the string-rewriting system S, and then $A\cdot S\cdot A$ is simply defined as the free abelian group that is generated by the set $\{a\cdot[e]\cdot b\mid a,b\in M, e\in S\}$ with a left-action defined by $c(a\cdot[e]\cdot b)=ca\cdot[e]\cdot b$ and a right-action defined by $(a\cdot[e]\cdot b)d=a\cdot[e]\cdot bd$ for all $a,b,c,d\in M$ and $e\in S$. The exact definition of the A-bimodule Π and a description of the embedding of Π in $A\cdot S\cdot A$ can also be found in [16]. Further, it turned out that a homotopy base $B\subseteq P^{(2)}(\Gamma(\Sigma;S))$ yields a set of generators for the A-bimodule Π [25,26]. It follows that $(\Sigma;S)$ has FHT, if it has FDT [27].

The operation of forming the *tensor product* $M_r\otimes_A M_\ell$ of a right A-module M_r and a left A-module M_ℓ yields an abelian group $G:=M_r\otimes_A M_\ell$. This group is the factor group of the free abelian group generated by $M_r\times M_\ell$ with respect to the subgroup specified by the following equations, where $a,a'\in M_r$, $b,b'\in M_\ell$, and $c\in A$:

$$((a+a'),b)=(a,b)+(a',b),\ (a,(b+b'))=(a,b)+(a,b') \quad \text{and}$$
$$(ac,b)=(a,cb).$$

Thus, G is uniquely determined by M_r and M_ℓ up to isomorphisms. If, in addition, M_r is an A-bimodule, then G inherits the structure of a left A-module, and if M_ℓ is an A-bimodule, then G inherits the structure of a right A-module.

As Π is an A-bimodule, and $\mathbb{Z}$ is the trivial left (right) A-module, we see that $\Pi^{(\ell)}:=\Pi\otimes_A\mathbb{Z}$ is a left A-module, and $\Pi^{(r)}:=\mathbb{Z}\otimes_A\Pi$ is a right A-module. From the defining equations of the tensor product and the fact that A acts trivially on $\mathbb{Z}$ we see that by forming the tensor product $\Pi\otimes_A\mathbb{Z}$ we simply trivialize the right action of A

on Π, and analogously, by forming the tensor product $\mathbb{Z}\otimes_A \Pi$ we trivialize the left action of A on Π. The embedding of Π in $A\cdot S\cdot A$ induces an embedding of $\Pi^{(\ell)}$ in $A\cdot S$ (of $\Pi^{(r)}$ in $S\cdot A$) [16], where $A\cdot S$ ($S\cdot A$) denotes the free left (right) A-module that is generated by S. Actually these embeddings extend to exact sequences of free left (right) A-modules of the form

$$0 \to \Pi^{(\ell)} \to A\cdot S \to A\cdot\Sigma \to A \to \mathbb{Z} \to 0,$$
$$0 \to \Pi^{(r)} \to S\cdot A \to \Sigma\cdot A \to A \to \mathbb{Z} \to 0,$$

where $A\cdot\Sigma$ ($\Sigma\cdot A$) denotes the free left (right) A-module that is generated by Σ (see, e.g., [18] for a detailed description of the morphisms $A\cdot S\to A\cdot\Sigma\to A$). Hence, we see that a finitely presented monoid M is left-FP_3 (right-FP_3) if $\Pi^{(\ell)}$ ($\Pi^{(r)}$) is finitely generated as a left (right) A-module. It follows that the property FHT implies the homological finiteness conditions left- and right-FP_3, as $\Pi^{(\ell)}$ and $\Pi^{(r)}$ are finitely generated, if Π is. Actually, it is a consequence of a generalization of Schanuel's lemma (see, e.g., [5, p. 193]) that $\Pi^{(\ell)}$ ($\Pi^{(r)}$) is finitely generated as a left (right) A-module if M is left-FP_3 (right-FP_3), that is, a finitely presented monoid M is left-FP_3 (right-FP_3) if and only if $\Pi^{(\ell)}$ ($\Pi^{(r)}$) is finitely generated as a left (right) A-module.

For future reference we need the mapping $\partial : P(\Gamma(\Sigma;S))\to A\cdot S$ that is defined as follows: if $p=x_1e_1^{\varepsilon_1}y_1\circ x_2e_2^{\varepsilon_2}y_2\circ\cdots\circ x_ne_n^{\varepsilon_n}y_n$ is a path in $\Gamma(\Sigma;S)$, where $n\geqslant 1$, $x_i,y_i\in\Sigma^*$, $e_i\in S$, $\varepsilon_i\in\{\pm1\}$, $i=1,\dots,n$, and $\circ$ denotes the composition of edges and paths, then

$$\partial(p) := \sum_{i=1}^{n} \varepsilon_i\cdot\bar{x}_i[e_i].$$

Here $[e_i]$ is a formal generator corresponding to the element $e_i\in S$, and $\bar{x}_i$ denotes the element of M that is represented by the string x_i.

Let $B\subseteq P^{(2)}(\Gamma(\Sigma;S))$ be a homotopy base for $P^{(2)}(\Gamma(\Sigma;S))$. Then the image of $\Pi^{(\ell)}$ in $A\cdot S$ is generated by the set $\{\partial(p)-\partial(q)\mid (p,q)\in B\}$. Thus, modulo the embedding of $\Pi^{(\ell)}$ in $A\cdot S$, this set generates the left A-module $\Pi^{(\ell)}$.

For finitely presented monoids the relationships between the various finiteness conditions described above can be summarized by the following implications, where FCP denotes the property of admitting a finite convergent presentation:

$$\mathsf{FCP} \;\Rightarrow\; \mathsf{FDT} \;\Rightarrow\; \mathsf{FHT} \;\Rightarrow\; \text{left-}\mathsf{FP}_3 \quad\text{and}\quad \text{right-}\mathsf{FP}_3$$

and

$$\mathsf{FCP} \;\Rightarrow\; \text{left-}\mathsf{FP}_\infty \;\Rightarrow\; \text{left-}\mathsf{FP}_n\ (n\geqslant 4) \;\Rightarrow\; \text{left-}\mathsf{FP}_3$$

and

$$\mathsf{FCP} \;\Rightarrow \text{right-}\mathsf{FP}_\infty \;\Rightarrow\; \text{right-}\mathsf{FP}_n\ (n\geqslant 4) \;\Rightarrow\; \text{right-}\mathsf{FP}_3.$$

In passing we would like to point out that it is known that apart from the implication '$\mathsf{FDT}\Rightarrow\mathsf{FHT}$' none of the above implications can be reversed (see, e.g., [15]), while it is still open whether FHT implies FDT for finitely presented monoids. [3] Because of the implications above Theorem 4.1 yields the following undecidability results, where part (5) follows by symmetry.

Corollary 4.2. *For the class of finitely presented monoids with linear-time decidable word problem the following properties are undecidable in general*:
(1) *The homological finiteness conditions left*-FP_∞ *and left*-FP_n *for all* $n \geqslant 3$.
(2) *The property of having finite homology type* FHT.
(3) *The property of having finite derivation type* FDT.
(4) *The existence of a finite convergent presentation.*
(5) *The homological finiteness conditions right*-FP_∞ *and right*-FP_n *for all* $n \geqslant 3$.

It remains to prove Theorem 4.1. For that we will use the following example monoid M that is taken from [17,18]. The monoid M is given through the finite presentation $(\Sigma; R)$, where $\Sigma := \{a, b, c, d, e\}$ and

$$R := \{ab \to a, da \to ac, ea \to ac\}.$$

By choosing the weight 1 for a, b, and c and the weight 2 for d and e, we see that R is weight-reducing. However, it is not confluent, as the critical pairs (acb, da) and (acb, ea), that result from overlapping the rules $da \to ac$ and $ea \to ac$ with the first rule, are not resolvable. However, R is equivalent to the infinite convergent system

$$R^\infty := R \cup \{ac^n b \to ac^n \mid n \geqslant 1\},$$

which is weight-reducing and left-regular. In the following lemma we summarize the important properties of this monoid.

Lemma 4.3. *The monoid M presented by $(\Sigma; R)$ has the following properties*:
(1) *Its word problem is decidable in linear time*, *in fact*, *normal forms with respect to the convergent system R^∞ can be computed in linear time.*
(2) *It is neither left- nor right*-FP_3.

Proof. In [20] Ó'Dúnlaing shows how to compute irreducible descendants with respect to left-regular string-rewriting systems in linear time. He only considers length-reducing left-regular systems S for which range(S) is finite, but it is clear that his construction works for weight-reducing systems as well. The system R^∞ has an infinite set of right-hand sides, but as there is a very close correspondence between the left-hand side and the right-hand side of each rule of R^∞, his algorithm is easily adopted to this particular system.

In [18] it is shown that the monoid M is not left-FP_3, and in [16] it is shown that M is not right-FP_3, either. □

[3] Recently S. Pride and F. Otto have presented an example of a finitely presented monoid that is FHT but not FDT.

By combining the example monoid M above with the monoid $N_{1,w}$ of [13] we now provide a proof for Theorem 4.1.

Proof of Theorem 4.1. Let $(\Delta_1; T_{1,w})$ be the finite presentation of the monoid $N_{1,w}$ constructed in [13] (see the end of Section 2), and let M be the example monoid with the presentation $(\Sigma; R)$ considered above. We assume that the alphabets Δ_1 and Σ are disjoint. Further, we choose two additional letters α, β, and take $\Omega := \Sigma \cup \Delta_1 \cup \{\alpha, \beta\}$. We define a finitely presented monoid E_w as an extension of the free product $M * N_{1,w}$. The monoid E_w is given through the presentation $(\Omega; S_w)$, where

$$S_w := R \cup T_{1,w} \cup \{\alpha HE\beta \to cb, \alpha O\beta \to c\}.$$

Claim 1. *If $w \in L$, then S_w is equivalent to a finite convergent string-rewriting system.*

Proof. If $w \in L$, then the subsystem $T_{1,w}$ of S_w is equivalent to a finite convergent system $T_{1,w}^{\infty}$, and $HE \leftrightarrow^*_{T_{1,w}} O$. Hence, it is easily seen that the system S_w is equivalent to the finite string-rewriting system

$$\hat{S}_w := R \cup T_{1,w}^{\infty} \cup \{\alpha O\beta \to c, cb \to c\}.$$

The subsystem $R \cup \{cb \to c\}$ is weight-reducing and confluent, as all its critical pairs resolve. The subsystems $T_{1,w}^{\infty}$ and $R \cup \{cb \to c\}$ have no letter in common, and as they both are convergent, their union is convergent, as convergence is a modular property of string-rewriting systems [21]. Further, the rule $\alpha O\beta \to c$ is the only one involving the letters α and β. Hence, in any $\hat{S}_w$-reduction sequence this rule can be used only a finite number of times. Thus, the system $\hat{S}_w$ is noetherian, and as the rule $\alpha O\beta \to c$ does not overlap with any other rule, we see that the system $\hat{S}_w$ is convergent. □

Claim 2. *If $w \in L$, then the word problem for the monoid E_w is decidable in linear time.*

Proof. It suffices to prove that there is an algorithm that, given a string $u \in \Omega^*$ as input, determines the normal form of $u \bmod \hat{S}_w$ in linear time. From [13, Lemma 4.6], we know that the normal form computation $\bmod T_{1,w}^{\infty}$ can be performed in linear time. Further, the subsystem $R_1 := R \cup \{cb \to c\}$ is weight-reducing and confluent, and therewith a string $v \in \Sigma^*$ can be reduced in linear time to its irreducible descendant $\bmod R_1$.

We now compute the normal form of a string $u \in \Omega^+ \bmod \hat{S}_w$ as follows, where we proceed in three stages:

(1) First each syllable $v \in \Delta_1^+$ is replaced by its normal form $\hat{v} \bmod T_{1,w}^{\infty}$. This step is performed in linear time. As $T_{1,w}^{\infty}$ does not contain any special rules, we have $\hat{v} \neq \lambda$ for each of these syllables.
(2) The rule $\alpha O\beta \to c$ is applied as long as possible. Certainly, the number of these applications is bounded from above by the number $|u|_{\alpha}$. The resulting string is irreducible $\bmod T_{1,w}^{\infty} \cup \{\alpha O\beta \to c\}$.
(3) Each syllable $x \in \Sigma^+$ is reduced to its irreducible descendant $\hat{x} \bmod R_1$. Again this only takes linear time. Further, as R_1 contains no special rules either, we have

$\hat{x} \neq \lambda$ for each of these syllables. Thus, the resulting string $\hat{u}$ is irreducible mod $\hat{S}_w$, that is, it is the normal form of u mod $\hat{S}_w$.

As each of the three steps above can be executed in linear time, we see that normal forms mod $\hat{S}_w$ can be computed in linear time. This in turn implies that the word problem for the monoid E_w is decidable in linear time. □

It follows that the monoid E_w is a member of the class $\mathscr{C}_{\text{lin}}$ and that it has property P, whenever the string w belongs to the language L. For the remaining part of the proof we assume that w does not belong to the language L. In this case the monoid E_w has the infinite presentation $(\Omega; S_w^\infty)$, where

$$S_w^\infty := R^\infty \cup T_{1,w}^\infty \cup \{\alpha HE\beta \to cb, \alpha O\beta \to c\}.$$

Here $T_{1,w}^\infty$ denotes the infinite convergent system that is equivalent to $T_{1,w}$.

Claim 3. *The system S_w^∞ is convergent.*

Proof. The subsystems R^∞ and $T_{1,w}^\infty$ are both convergent, and they have no letter in common. Thus, their union is convergent. Further, the rules $\alpha HE\beta \to cb$ and $\alpha O\beta \to c$ are the only rules containing the letters α and β. Hence, an S_w^∞-reduction sequence can only contain a finite number of applications of these two rules. As the system $R^\infty \cup T_{1,w}^\infty$ is noetherian, it follows that S_w^∞ is noetherian. Finally, as by our assumption $w \notin L$, we know from [13] that HE and O are irreducible mod $T_{1,w}^\infty$, and hence, the two rules $\alpha HE\beta \to cb$ and $\alpha O\beta \to c$ do not overlap with any rules from S_w^∞. Thus, the system S_w^∞ is convergent. □

As the system S_w^∞ contains the system R^∞ as a subsystem, and as no rule from the difference $S_w^\infty \setminus R^\infty$ is applicable to a string from Σ^*, Claim 3 implies in particular that the monoid M is embedded in the monoid E_w by the identity mapping on Σ^*.

Next we will show that Claim 2 also holds for the case that $w \notin L$.

Claim 4. *If $w \notin L$, then the word problem for the monoid E_w is decidable in linear time.*

Proof. From [13, Lemma 4.6], we know that the normal form computation mod $T_{1,w}^\infty$ can be performed in linear time, and from Lemma 4.3(1) we see that normal forms mod R^∞ can be computed in linear time. Also the rules $\alpha HE\beta \to cb$ and $\alpha O\beta \to c$ are the only ones that involve the letters α and β. Hence, we can proceed as in the proof of Claim 2, that is, given a string $u \in \Omega^*$, we can determine the normal form of u mod S_w^∞ in linear time. □

Thus, the monoid E_w belongs to the class $\mathscr{C}_{\text{lin}}$ also in the case that $w \notin L$. It remains to prove that in this situation E_w does not have property P. For this it suffices to establish the following claim.

Claim 5. *If $w \notin L$, then the monoid E_w is not left-*FP_3.

Proof. As observed above E_w is presented by the infinite convergent string-rewriting system

$$S_w^\infty = R^\infty \cup T_{1,w}^\infty \cup \{\alpha HE\beta \to cb, \alpha O\beta \to c\}.$$

Let $\Gamma_1^\infty := \Gamma(\Sigma; R^\infty)$, $\Gamma_2^\infty := \Gamma(\Delta_1; T_{1,w}^\infty)$, and $\Gamma^\infty := \Gamma(\Omega; S_w^\infty)$ be the infinite graphs that are associated with the presentations $(\Sigma; R^\infty)$, $(\Delta_1; T_{1,w}^\infty)$, and $(\Omega; S_w^\infty)$, respectively. Obviously Γ_1^∞ and Γ_2^∞ are subgraphs of Γ^∞. Further, let $\Gamma_1 := \Gamma(\Sigma; R)$, $\Gamma_2 := \Gamma(\Delta_1; T_{1,w})$, and $\Gamma := \Gamma(\Omega; S_w)$ be the graphs that are associated with the finite presentations $(\Sigma; R)$, $(\Delta_1; T_{1,w})$, and $(\Omega; S_w)$, respectively. Then Γ_1 and Γ_2 are subgraphs of Γ. As R and R^∞ are equivalent, for each rule $(\ell, r) \in R^\infty \smallsetminus R$, there is a path $p_{(\ell,r)} \in P(\Gamma_1)$ such that $p_{(\ell,r)}$ leads from ℓ to r. Also $T_{1,w}$ and $T_{1,w}^\infty$ are equivalent, and so for each rule $(\ell, r) \in T_{1,w}^\infty \smallsetminus T_{1,w}$, there is a path $p_{(\ell,r)} \in P(\Gamma_2)$ such that $p_{(\ell,r)}$ leads from ℓ to r. By mapping each vertex $v \in \Omega^*$ onto itself, by mapping each edge corresponding to a rule of $R \cup T_{1,w}$ onto itself, and by mapping each edge $e = (u; \ell, r; v)$, where $u, v \in \Omega^*$ and $(\ell, r) \in R^\infty \smallsetminus R$ or $(\ell, r) \in T_{1,w}^\infty \smallsetminus T_{1,w}$, onto the path $u p_{(\ell,r)} v$, a morphism $\varphi : \Gamma^\infty \to \Gamma$ of graphs in the sense of [29] is obtained. Notice that the path $u p_{(\ell,r)} v$ is simply the path in Γ that is obtained from the path $p_{(\ell,r)}$ by concatenating each vertex and each edge of $p_{(\ell,r)}$ with the string u from the left and with the string v from the right.

By restricting the morphism φ to the subgraphs Γ_1^∞ and Γ_2^∞, respectively, we obtain corresponding morphisms $\varphi_1 : \Gamma_1^\infty \to \Gamma_1$ and $\varphi_2 : \Gamma_2^\infty \to \Gamma_2$.

The critical pairs of S_w^∞ are just those of R^∞, which can be resolved mod R^∞, and those of $T_{1,w}^\infty$, which can be resolved mod $T_{1,w}^\infty$. For each critical pair (e_1, e_2) of edges corresponding to a critical pair of R^∞, we fix a pair of positive paths $p_1, p_2 \in P_+(\Gamma_1^\infty)$ such that $(e_1 \circ p_1, e_2 \circ p_2) \in P^{(2)}(\Gamma_1^\infty)$. By C_1' we denote the subset

$$C_1' := \{(e_1 \circ p_1, e_2 \circ p_2) \mid (e_1, e_2) \text{ is a critical pair of } R^\infty\} \subseteq P^{(2)}(\Gamma_1^\infty).$$

Analogously, for each critical pair (e_3, e_4) of edges corresponding to a critical pair of $T_{1,w}^\infty$, we fix a pair of positive paths $p_3, p_4 \in P_+(\Gamma_2^\infty)$ such that $(e_3 \circ p_3, e_4 \circ p_4) \in P^{(2)}(\Gamma_2^\infty)$. By C_2' we denote the subset

$$C_2' := \{(e_3 \circ p_3, e_4 \circ p_4) \mid (e_3, e_4) \text{ is a critical pair of } T_{1,w}^\infty\} \subseteq P^{(2)}(\Gamma_2^\infty).$$

Then $C_1' \cup C_2'$ is a homotopy base for Γ^∞, that is, the only homotopy relation $\sim$ on $P(\Gamma^\infty)$ containing $C_1' \cup C_2'$ is the set $P^{(2)}(\Gamma^\infty)$ itself ([29, Theorem 5.2]). As Γ is a subgraph of Γ^∞, it follows from [29, Corollary 3.7] that

$$B_1' := \{(\varphi_1(p), \varphi_1(q)) \mid (p, q) \in C_1'\} \subseteq P^{(2)}(\Gamma_1)$$

together with

$$B_2' := \{(\varphi_2(p), \varphi_2(q)) \mid (p, q) \in C_2'\} \subseteq P^{(2)}(\Gamma_2)$$

forms a homotopy base for $P^{(2)}(\Gamma)$.

Let A_1, A_2, and A denote the integral monoid rings $\mathbb{Z}M$, $\mathbb{Z}N_{1,w}$, and $\mathbb{Z}E_w$, respectively, and let $\Pi_1^{(\ell)}$, $\Pi_2^{(\ell)}$, and $\Pi^{(\ell)}$ be the left A_1-, A_2- and A-modules corresponding to the monoids M, $N_{1,w}$, and E_w, respectively. Then

$$\partial(B_1') := \{\partial(p) - \partial(q) \mid (p, q) \in B_1'\} \subseteq A_1 \cdot R$$

generates the image of $\Pi_1^{(\ell)}$ in $A_1 \cdot R$,

$$\partial(B_2') := \{\partial(p) - \partial(q) \mid (p,q) \in B_2'\} \subseteq A_2 \cdot T_{1,w}$$

generates the image of $\Pi_2^{(\ell)}$ in $A_2 \cdot T_{1,w}$, and

$$\partial(B_1' \cup B_2') := \{\partial(p) - \partial(q) \mid (p,q) \in B_1' \cup B_2'\} \subseteq A \cdot S_w$$

generates the image of $\Pi^{(\ell)}$ in $A \cdot S_w$.

Now in order to prove Claim 5 by contradiction we assume that the monoid E_w is left-FP_3. As observed before this is equivalent to the assumption that the left A-module $\Pi^{(\ell)}$ is finitely generated. Let C be a finite set of generators of $\Pi^{(\ell)}$. As $\partial(B_1' \cup B_2')$ is also a set of generators of $\Pi^{(\ell)}$, each element of C can be expressed as a finite combination of elements of $\partial(B_1' \cup B_2')$. Thus, we see that the set of generators $\partial(B_1' \cup B_2')$ contains a finite subset that already generates the left A-module $\Pi^{(\ell)}$, that is, there exist finite subsets $B_1 \subseteq \partial(B_1') \subseteq A_1 \cdot R$ and $B_2 \subseteq \partial(B_2') \subseteq A_2 \cdot T_{1,w}$ such that $B_1 \cup B_2$ generates the left A-module $\Pi^{(\ell)}$.

For deriving the intended contradiction we will need the following observation.

Claim 6. *For all $x \in E_w \setminus M$ and all $y \in M$, the product xy belongs to $E_w \setminus M$.*

Proof. Let $x \in \Omega^*$ be a string that does not represent an element of the submonoid M, that is, x is not congruent $\bmod S_w$ to any string from Σ^*. Then the normal form $\hat{x}$ of x $\bmod S_w^\infty$ contains letters from $\Delta_1 \cup \{\alpha, \beta\}$, that is, $\hat{x} = x_1 \eta x_2$ for some $x_1 \in \Omega^*$, $\eta \in \Delta_1 \cup \{\alpha, \beta\}$, and $x_2 \in \Sigma^*$.

For $y \in \Sigma^*$, the normal form $\hat{y}$ of y $\bmod S_w^\infty$ is an element of Σ^*. Hence, the product xy is congruent to $\hat{x}\hat{y} = x_1 \eta x_2 \hat{y}$. As the left-hand side of no rule of S_w^∞ contains occurrences of letters from Σ as well as from $\Delta_1 \cup \{\alpha, \beta\}$, we see that the normal form of xy is simply the string $x_1 \eta x_0$, where x_0 is the irreducible descendant of $x_2 y$ $\bmod R^\infty$. It follows that xy is not congruent to any string from Σ^*, and so it does not represent an element of the submonoid M of E_w. $\square$

The proof of Claim 5 will now be completed by establishing the following claim.

Claim 7. *The set B_1 generates the left A_1-module $\Pi_1^{(\ell)}$.*

Proof. Let $p \in A_1 \cdot R$ such that p belongs to $\Pi_1^{(\ell)}$. As $\Pi_1^{(\ell)}$ is generated by $\partial(B_1')$, it is contained in $\Pi^{(\ell)}$, which is generated by $\partial(B_1' \cup B_2')$. By our assumption the latter is also generated by $B_1 \cup B_2$. Hence, we see that p can be written as a linear combination of elements of B_1 and B_2 in $A \cdot S_w$. However, as $p \in A_1 \cdot R$ and $B_2 \subseteq A_2 \cdot T_{1,w}$, and as S_w, which includes the union $R \cup T_{1,w}$, freely generates the left A-module $A \cdot S_w$, it follows that p is actually contained in the submodule of $A \cdot S_w$ that is generated by B_1. Thus, p can be written as a finite sum $p = \sum_{\substack{m \in E_w \\ b \in B_1}} z_{m,b} \cdot m \cdot b$, where $z_{m,b} \in \mathbb{Z}$, and only finitely many $z_{m,b}$ are not equal to 0. As $M \subseteq E_w$, we can decompose this representation of p

as $p = p_1 + p_2$, where

$$p_1 := \sum_{\substack{m\in M\\ b\in B_1}} z_{m,b} \cdot m \cdot b \quad \text{and} \quad p_2 := \sum_{\substack{m\in E_w\setminus M\\ b\in B_1}} z_{m,b} \cdot m \cdot b.$$

As $B_1 \subseteq A_1 \cdot R$, it follows that $p_1 \in A_1 \cdot R$. On the other hand, for each $m \in E_w \setminus M$ and each element $b \in B_1$, all the monoid elements occurring in the product $m \cdot b$ belong to $E_w \setminus M$ by Claim 6. Thus, as $p \in A_1 \cdot R$ and $p_1 \in A_1 \cdot R$, it follows that $p_2 = 0$, that is, $p = p_1$, which means that p belongs to the submodule of $A_1 \cdot R$ that is generated by B_1. Hence, B_1 does indeed generate the left A_1-module $\Pi_1^{(\ell)}$. □

As B_1 is a finite set, Claim 7 implies that the left A_1-module $\Pi_1^{(\ell)}$ is finitely generated. This in turn means that the monoid M is left-FP_3, which contradicts Lemma 4.3(2). Hence, the monoid E_w is not left-FP_3. This completes the proof of Claim 5. □

We see from Claims 2 and 4 that the finitely presented monoid E_w belongs to the class $\mathscr{C}_{\text{lin}}$, no matter whether or not w belongs to the language L. Further, Claim 1 shows that E_w has property P, if $w \in L$, and Claim 5 implies that E_w does not have property P, if $w \notin L$. As L is non-recursive, this shows that property P is undecidable for the class $\mathscr{C}_{\text{lin}}$, which completes the proof of Theorem 4.1. □

5. Concluding remarks

In [13] the first two authors have shown that all linear Markov properties are undecidable for the class $\mathscr{C}_{\text{lin}}$ by presenting a construction that allows to carry over Markov's undecidability proof from the class of all finitely presented monoids to the class $\mathscr{C}_{\text{lin}}$. Here we have used this construction to prove that also many other properties of monoids that are not known to be (linear) Markov properties are undecidable for the class $\mathscr{C}_{\text{lin}}$. We close this paper by presenting still another undecidability result.

Let P_z denote the property of monoids to have a zero. Obviously, each finitely generated free monoid is an example of a monoid from the class $\mathscr{C}_{\text{lin}}$ that does not have a zero, while the presentation $(a, z; \{az \to z, za \to z, zz \to z\})$ gives a monoid M_z from $\mathscr{C}_{\text{lin}}$ with a zero. If a monoid M given by a presentation $(\Sigma; S)$ does not have a zero, then by adding a letter z and the rules $za \to z$ and $az \to z$ for all $a \in \Sigma \cup \{z\}$, we obtain a monoid M' with a zero such that M is embedded in M' by the identity mapping on Σ^*. On the other hand, if a monoid M given by a presentation $(\Sigma; S)$ does have a zero, then by forming the free product $M * F_1$ of the monoid M and the free monoid F_1 of rank one we obtain a monoid M' without a zero, and again M is embedded in M' by the identity mapping on Σ^*. Even more, in each of these two cases M' belongs to the class $\mathscr{C}_{\text{lin}}$, if M does. Thus, we see that the property P_z is not a (linear) Markov property, nor is its negation a (linear) Markov property. Hence, neither the result of [13] nor that of [19] is applicable to this property. Nevertheless, we can easily derive the following undecidability result.

Theorem 5.1. *The property of having a zero is undecidable for the class of finitely presented monoids with linear-time decidable word problem.*

Proof. Let M_z be the aforementioned monoid with a zero from the class $\mathscr{C}_{\text{lin}}$. Further, let N_w be again the monoid constructed in [13]. We consider the free product $Z_w := M_z * N_w$. As in the given presentation of M_z no non-empty string is congruent to the empty string, it can be shown similarly to the proof of Claim 2 (within the proof of Theorem 4.1) that Z_w has word problem decidable in linear time. Thus, Z_w belongs to the class $\mathscr{C}_{\text{lin}}$.

If the string w belongs to the language L, then N_w is the trivial monoid, and so Z_w is isomorphic to the monoid M_z, that is, Z_w has a zero. On the other hand, if w is not in L, then N_w is non-trivial, and so Z_w is a non-trivial free product, and as such it does not have a zero. It follows that the property P_z is undecidable for $\mathscr{C}_{\text{lin}}$. $\square$

The same reasoning applies to the property of having a left-zero and the property of having a right-zero. Thus, also these properties are undecidable for the class $\mathscr{C}_{\text{lin}}$.

Acknowledgements

We are grateful to the referees whose questions and detailed comments helped to greatly improve the presentation of this paper.

References

[1] J. Avenhaus, K. Madlener, Subrekursive Komplexität bei Gruppen: I. Gruppen mit vorgeschriebener Komplexität, Acta Inform. 9 (1977) 87–104.
[2] J. Berstel, Transductions and Context-free Languages, Teubner, Stuttgart, 1979.
[3] R.V. Book, Confluent and other types of Thue systems, J. Assoc. Comput. Mach. 29 (1982) 171–182.
[4] R.V. Book, F. Otto, String-Rewriting Systems, Springer, New York, 1993.
[5] K.S. Brown, Cohomology of Groups, Springer, New York, 1982.
[6] C.M. Campbell, E.F. Robertson, N. Ruškuc, R.M. Thomas, Automatic semigroups, Theoret. Comput. Sci. 250 (2001) 365–391.
[7] D.E. Cohen, String rewriting and homology of monoids, Math. Struct. Comput. Sci. 7 (1997) 207–240.
[8] R. Cremanns, F. Otto, FP_∞ is undecidable for finitely presented monoids with word problems decidable in polynomial time, Mathematische Schriften Kassel 11/98, Universität Kassel, September 1998.
[9] D.B.A. Epstein, J.W. Cannon, D.F. Holt, S.V.F. Levy, M.S. Paterson, W.P. Thurston, Word Processing In Groups, Jones and Bartlett, Boston, 1992.
[10] V.S. Guba, M. Sapir, Diagram groups, Mem. Amer. Math. Soc. 130 (1997) 1–117.
[11] J.F.P. Hudson, Regular rewrite systems and automatic structures, in: J. Almeida, G.M.S. Gomes, P.V. Silva (Eds.), Semigroups, Automata and Languages, World Scientific, Singapore, 1996, pp. 145–152.
[12] G. Huet, D. Lankford, On the uniform halting problem for term rewriting systems, Laboratory Report No. 283, INRIA, Le Chesnay, France, March 1978.
[13] M. Katsura, Y. Kobayashi, Undecidable properties of monoids with word problem solvable in linear time, Theoret. Comput. Sci. 290 (2003) 1301–1316.
[14] D.E. Knuth, P. Bendix, Simple word problems in universal algebras, in: J. Leech (Ed.), Computational Problems in Abstract Algebra, Pergamon Press, New York, 1970, pp. 263–297.
[15] Y. Kobayashi, F. Otto, On homological and homotopical finiteness conditions for finitely presented monoids, Internat. J Algebra Comput. 11 (2001) 391–403.
[16] Y. Kobayashi, F. Otto, Some exact sequences for the homotopy (bi-) module of a monoid, Internat. J. Algebra Comput. 12 (2002) 247–284.

[17] Y. Lafont, A finiteness condition for monoids presented by complete rewriting systems (after C. C. Squier), J. Pure and Appl. Algebra 98 (1995) 229–244.
[18] Y. Lafont, A. Prouté, Church-Rosser property and homology of monoids, Math. Struct. Comput. Sci. 1 (1991) 297–326.
[19] A. Markov, Impossibility of algorithms for recognizing some properties of associative systems, Dokl. Akad. Nauk SSSR 77 (1951) 953–956.
[20] C. Ó'Dúnlaing, Infinite regular Thue systems, Theoret. Comput. Sci. 25 (1983) 171–192.
[21] F. Otto, Modular properties of monoids and string-rewriting systems, in: Ch. Nehaniv, M. Ito (Eds.), Algebraic Engineering, World Scientific, Singapore, 1999, pp. 538–554.
[22] F. Otto, M. Katsura, Y. Kobayashi, Infinite convergent string-rewriting systems and cross-sections for finitely presented monoids, J. Symbolic Comput. 26 (1998) 621–648.
[23] F. Otto, A. Sattler-Klein, The property FDT is undecidable for finitely presented monoids that have polynomial-time decidable word problems, Internat. J. Algebra Comput. 10 (2000) 285–307.
[24] F. Otto, A. Sattler-Klein, K. Madlener, Automatic monoids versus monoids with finite convergent presentations, in: T. Nipkow (Ed.), Rewriting Techniques and Applications, Proc. RTA'98, Lecture Notes in Computer Science, Vol. 1379, Springer, Berlin, 1998, pp 32–46.
[25] S.J. Pride, Low-dimensional homotopy theory for monoids, Internat. J. Algebra Comput. 5 (1995) 631 –649.
[26] S.J. Pride, Geometric methods in combinatorial semigroup theory, in: J. Fountain (Ed.), Proc. Internat. Conf. on Semigroups, Formal Languages, and Groups, Kluwer, Dordrecht, 1995, pp. 215–232.
[27] S.J. Pride, J. Wang, Rewriting systems, finiteness conditions, and associated functions, in: J.C. Birget, S. Margolis, J. Meakin, M. Sapir (Eds.), Algorithmic Problems in Groups and Semigroups, Birkhäuser, Boston, 2000, pp. 195–216.
[28] A. Sattler-Klein, New undecidability results for finitely presented monoids, in: H. Comon (Ed.), Rewriting Techniques and Applications, Proc. RTA'97, Lecture Notes in Computer Science, Vol. 1232, Springer, Berlin, 1997, pp. 68–82.
[29] C.C. Squier, A finiteness condition for rewriting systems. Revision by F. Otto and Y. Kobayashi, Theoret. Comput. Sci. 131 (1994) 271–294.
[30] X. Wang, S.J. Pride, Second order Dehn functions of groups and monoids, Internat. J. Algebra Comput. 10 (2000) 425–456.